Spaceability in Banach and quasi-Banach sequence spaces

G. Botelho*, D. Diniz, V. V. Fávaro†, D. Pellegrino‡

Abstract

Let X be a Banach space. We prove that, for a large class of Banach or quasi-Banach spaces E of X-valued sequences, the sets $E - \bigcup_{q\in\Gamma} \ell_q(X)$, where Γ is any subset of $(0,\infty]$, and $E - c_0(X)$ contain closed infinite-dimensional subspaces of E (if non-empty, of course). This result is applied in several particular cases and it is also shown that the same technique can be used to improve a result on the existence of spaces formed by norm-attaining linear operators.

Introduction

A subset A of a Banach or quasi-Banach space E is *μ-lineable* (*spaceable*) if $A \cup \{0\}$ contains a μ-dimensional (closed infinite-dimensional) linear subspace of E. The last few years have witnessed the appearance of lots of papers concerning lineability and spaceability (see, for example, [1, 2, 3, 10, 13]). The aim of this paper is to explore a technique to prove lineability and spaceability that can be applied in several different settings. It is our opinion that this technique was first used in the context of lineability/spaceability in our preprint [4], of which this paper is an improved version.

Let c denote the cardinality of the set of real numbers $\mathbb{R}$. In [10] it is proved that $\ell_p - \ell_q$ is c-lineable for every $p > q \geq 1$. With the help of [7] this result can be substantially improved in the sense that $\ell_p - \bigcup_{1\leq q<p} \ell_q$ is spaceable for every $p > 1$. In this paper we address the following questions: What about the non-locally convex range $0 < p < 1$? Can these results be generalized to sequence spaces other than ℓ_p?

*Supported by CNPq Grant 306981/2008-4 and INCT-Matemática.

†Supported by Fapemig Grant CEX-APQ-00208-09.

‡Supported by INCT-Matemática, PROCAD-NF-Capes, CNPq Grants 620108/2008-8 (Ed. Casadinho) and 301237/2009-3.

2010 Mathematics Subject Classification: 46A45, 46A16, 46B45.

As to the first question, it is worth recalling that the structure of quasi-Banach spaces (or, more generally, metrizable complete tvs, called F-spaces) is quite different from the structure of Banach spaces. For our purposes, the consequence is that the extension of lineability/spaceability arguments from Banach to quasi-Banach spaces is not straightforward in general. For example, in [14, Section 6] it is essentially proved (with a different terminology) that if Y is a closed infinite-codimensional linear subspace of the Banach space X, then $X - Y$ is spaceable. A counterexample due to Kalton [5, Theorem 1.1] shows that this result is not valid for quasi-Banach spaces (there exists a quasi-Banach space K with an 1-dimensional subspace that is contained in all closed infinite-dimensional subspaces of K). Besides, the search for closed infinite-dimensional subspaces of quasi-Banach spaces is a quite delicate issue. Even fundamental facts are unknown, for example the following problem is still open (cf. [6, Problem 3.1]): Does every (infinite-dimensional) quasi-Banach space have a proper closed infinite-dimensional subspace? Nevertheless we solve the first question in the positive: as a particular case of our results we get that $\ell_p - \bigcup_{1 \leq q < p} \ell_q$ is spaceable for every $p > 0$ (cf. Corollary 1.7).

As to the second question, we identify a large class of vector-valued sequence spaces, called *invariant sequence spaces* (cf. Definition 1.1), such that if E is an invariant Banach or quasi-Banach space of X-valued sequences, where X is a Banach space, then the sets $E - \bigcup_{q \in \Gamma} \ell_q(X)$, where Γ is any subset of $(0, \infty]$, and $E - c_0(X)$ are spaceable whenever they are non-empty (cf. Theorem 1.3). Several classical sequence spaces are invariant sequence spaces (cf. Example 1.2).

In order to make clear that the technique we use can be useful in a variety of other situations, we finish the paper with an application to the c-lineability of sets of norm-attaining linear operators (cf. Proposition 2.1).

From now on all Banach and quasi-Banach spaces are considered over a fixed scalar field $\mathbb{K}$ which can be either $\mathbb{R}$ or $\mathbb{C}$.

1 Sequence spaces

In this section we introduce a quite general class of scalar-valued or vector-valued sequence spaces and prove that certain of their remarkable subsets have spaceable complements.

Definition 1.1. Let $X \neq \{0\}$ be a Banach space.
(a) Given $x \in X^{\mathbb{N}}$, by x^0 we mean the zerofree version of x, that is: if x has only finitely many non-zero coordinates, then $x^0 = 0$; otherwise, $x^0 = (x_j)_{j=1}^{\infty}$ where x_j is the j-th non-zero coordinate of x.
(b) By an *invariant sequence space over* X we mean an infinite-dimensional

Banach or quasi-Banach space E of X-valued sequences enjoying the following conditions:
(b1) For $x \in X^{\mathbb{N}}$ such that $x^0 \neq 0$, $x \in E$ if and only if $x^0 \in E$, and in this case $\|x\| \leq K\|x^0\|$ for some constant K depending only on E.
(b2) $\|x_j\|_X \leq \|x\|_E$ for every $x = (x_j)_{j=1}^{\infty} \in E$ and every $j \in \mathbb{N}$.
An *invariant sequence space* is an invariant sequence space over some Banach space X.

Several classical sequence spaces are invariant sequence spaces:

Example 1.2. (a) Given a Banach space X, it is obvious that for every $0 < p \leq \infty$, $\ell_p(X)$ (absolutely p-summable X-valued sequences), $\ell_p^u(X)$ (unconditionally p-summable X-valued sequences) and $\ell_p^w(X)$ (weakly p-summable X-valued sequences) are invariant sequence spaces over X with their respective usual norms (p-norms if $0 < p < 1$). In particular, ℓ_p, $0 < p \leq \infty$, are invariant sequence spaces (over $\mathbb{K}$).
(b) The Lorentz spaces $\ell_{p,q}$, $0 < p < \infty$, $0 < q < \infty$ (see, e.g., [12, 13.9.1]). It is easy to see that these spaces are invariant sequence spaces (over $\mathbb{K}$): indeed, given $0 \neq x^0 \in \ell_{p,q}$, the non-increasing rearrangement of x coincides with that of x^0. So $\|x\|_{p,q} = \|x^0\|_{p,q} < \infty$.
(c) The Orlicz sequence spaces (see, e.g., [8, 4.a.1]). Let M be an Orlicz function and ℓ_M be the corresponding Orlicz sequence space. The condition $M(0) = 0$ makes clear that ℓ_M is an invariant sequence space (over $\mathbb{K}$). For the same reason, its closed subspace h_M is an invariant sequence space as well.
(d) Mixed sequence spaces (see, e.g., [12, 16.4]). Given $0 < p \leq s \leq \infty$ and a Banach space X, by $\ell_{m(s;p)}(X)$ we mean the Banach (p-Banach if $0 < p < 1$) space of all mixed (s,p)-summable sequences on X. It is not difficult to see that $\ell_{m(s;p)}(X)$ is an invariant sequence space over X.

Now we can prove our main result. Given an invariant sequence space E over the Banach space X, regarding both E and $\ell_p(X)$ as subsets of $X^{\mathbb{N}}$, we can talk about the difference $E - \ell_p(X)$ and related ones.

Theorem 1.3. *Let E be an invariant sequence space over the Banach space X. Then*
(a) *For every $\Gamma \subseteq (0, \infty]$, $E - \bigcup_{q \in \Gamma} \ell_q(X)$ is either empty or spaceable.*
(b) *$E - c_0(X)$ is either empty or spaceable.*

Proof. Put $A = \bigcup_{q\in\Gamma} \ell_q(X)$ in (a) and $A = c_0(X)$ in (b). Assume that $E - A$ is non-empty and choose $x \in E - A$. Since E is an invariant sequence space, $x^0 \in E$, and obviously $x^0 \notin A$. Writing $x^0 = (x_j)_{j=1}^{\infty}$ we have that $x^0 \in E - A$ and $x_j \neq 0$ for every j. Split $\mathbb{N}$ into countably many infinite

pairwise disjoint subsets $(\mathbb{N}_i)_{i=1}^{\infty}$. For every $i \in \mathbb{N}$ set $\mathbb{N}_i = \{i_1 < i_2 < \ldots\}$ and define

$$y_i = \sum_{j=1}^{\infty} x_j e_{i_j} \in X^{\mathbb{N}}.$$

Observe that $y_i^0 = x^0$ for every i. So $0 \neq y_i^0 \in E$ for every i. Hence each $y_i \in E$ because E is an invariant sequence space. Let us see that $y_i \notin A$: in (a) this occurs because $\|y_i\|_r = \|x^0\|_r = \|x\|_r$ for every $0 < r \leq \infty$ and in (b) because $\|x_j\| \nrightarrow 0$. Let K be the constant of condition 1.1(b1) and define $\tilde{s} = 1$ if E is a Banach space and $\tilde{s} = s$ if E is a s-Banach space, $0 < s < 1$. For $(a_j)_{j=1}^{\infty} \in \ell_{\tilde{s}}$,

$$\sum_{j=1}^{\infty} \|a_j y_j\|^{\tilde{s}} = \sum_{j=1}^{\infty} |a_j|^{\tilde{s}} \|y_j\|^{\tilde{s}} \leq K^{\tilde{s}} \sum_{j=1}^{\infty} |a_j|^{\tilde{s}} \left\|y_j^0\right\|^{\tilde{s}}$$

$$= K^{\tilde{s}} \left\|x^0\right\|^{\tilde{s}} \sum_{j=1}^{\infty} |a_j|^{\tilde{s}} = K^{\tilde{s}} \left\|x^0\right\|^{\tilde{s}} \left\|(a_j)_{j=1}^{\infty}\right\|_{\tilde{s}}^{\tilde{s}} < \infty.$$

Thus $\sum_{j=1}^{\infty} \|a_j y_j\| < \infty$ if E is a Banach space and $\sum_{j=1}^{\infty} \|a_j y_j\|^s < \infty$ if E is a s-Banach space, $0 < s < 1$. In both cases the series $\sum_{j=1}^{\infty} a_j y_j$ converges in E, hence the operator

$$T\colon \ell_{\tilde{s}} \longrightarrow E \quad , \quad T\left((a_j)_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty} a_j y_j$$

is well defined. It is easy to see that T is linear and injective. Thus $\overline{T(\ell_{\tilde{s}})}$ is a closed infinite-dimensional subspace of E. We just have to show that $\overline{T(\ell_{\tilde{s}})} - \{0\} \subseteq E - A$. Let $z = (z_n)_{n=1}^{\infty} \in \overline{T(\ell_{\tilde{s}})}$, $z \neq 0$. There are sequences $\left(a_i^{(k)}\right)_{i=1}^{\infty} \in \ell_{\tilde{s}}$, $k \in \mathbb{N}$, such that $z = \lim_{k\to\infty} T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right)$ in E. Note that, for each $k \in \mathbb{N}$,

$$T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} a_i^{(k)} y_i = \sum_{i=1}^{\infty} a_i^{(k)} \sum_{j=1}^{\infty} x_j e_{i_j} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i^{(k)} x_j e_{i_j}.$$

Fix $r \in \mathbb{N}$ such that $z_r \neq 0$. Since $\mathbb{N} = \bigcup_{j=1}^{\infty} \mathbb{N}_j$, there are (unique) $m, t \in \mathbb{N}$ such that $e_{m_t} = e_r$. Thus, for each $k \in \mathbb{N}$, the r-th coordinate of $T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right)$ is the number $a_m^{(k)} x_t$. Condition 1.1(b2) assures that convergence in E implies coordinatewise convergence, so

$$z_r = \lim_{k\to\infty} a_m^{(k)} x_t = x_t \cdot \lim_{k\to\infty} a_m^{(k)}.$$

It follows that $x_t \neq 0$. Hence $\lim_{k\to\infty} |a_m^{(k)}| = \frac{\|z_r\|}{\|x_t\|} \neq 0$. For $j, k \in \mathbb{N}$, the m_j-th coordinate of $T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right)$ is $a_m^{(k)} x_j$. Defining $\alpha_m = \frac{\|z_r\|}{\|x_t\|} \neq 0$,

$$\lim_{k\to\infty} \|a_m^{(k)} x_j\| = \lim_{k\to\infty} |a_m^{(k)}| \|x_j\| = \|x_j\| \cdot \lim_{k\to\infty} |a_m^{(k)}| = \alpha_m \|x_j\|$$

for every $j \in \mathbb{N}$. On the other hand, coordinatewise convergence gives $\lim_{k\to\infty} \|a_m^{(k)} x_j\| = \|z_{m_j}\|$, so $\|z_{m_j}\| = \alpha_m \|x_j\|$ for each $j \in \mathbb{N}$. Observe that m, which depends on r, is fixed, so the natural numbers $(m_j)_{j=1}^{\infty}$ are pairwise distinct (remember that $\mathbb{N}_m = \{m_1 < m_2 < \ldots\}$).
(a) As $x^0 \notin A$, we have $\|x^0\|_q = \infty$ for all $q \in \Gamma$. Assume first that $\infty \notin \Gamma$. In this case,

$$\|z\|_q^q = \sum_{n=1}^{\infty} \|z_n\|^q \geq \sum_{j=1}^{\infty} \left\|z_{m_j}\right\|^q = \sum_{j=1}^{\infty} \alpha_m^q \cdot \|x_j\|^q = \alpha_m^q \cdot \left\|x^0\right\|_q^q = \infty,$$

for all $q \in \Gamma$, proving that $z \notin \bigcup_{q\in\Gamma} \ell_q(X)$. If $\infty \in \Gamma$,

$$\|z\|_\infty = \sup_n \|z_n\| \geq \sup_j \|z_{m_j}\| = \alpha_m \cdot \sup_j \|x_j\| = \alpha_m \|x^0\|_\infty = \infty,$$

proving again that $z \notin \bigcup_{q\in\Gamma} \ell_q(X)$.
(b) As $x^0 \notin A$, we have $\|x_j\| \nrightarrow 0$. Since $(\|z_{m_j}\|)_{j=1}^{\infty}$ is a subsequence of $(\|z_n\|)_{n=1}^{\infty}$, $\|z_{m_j}\| = \alpha_m \|x_j\|$ for every j and $\alpha_m \neq 0$, it is clear that $\|z_n\| \nrightarrow 0$. Thus $z \notin c_0(X)$.

Therefore $z \notin A$ in both cases, so $\overline{T(\ell_{\tilde{s}})} - \{0\} \subseteq E - A$. $\square$

We list a few consequences.

When we write $F \subset E$ we mean that E contains F as a linear subspace and $E \neq F$. We are not asking neither E to contain an isomorphic copy of F nor the inclusion $F \hookrightarrow E$ to be continuous.

Corollary 1.4. *Let E be an invariant sequence space over $\mathbb{K}$.*
(a) *If $0 < p \leq \infty$ and $\ell_p \subset E$, then $E - \ell_p$ is spaceable.*
(b) *If $c_0 \subset E$, then $E - c_0$ is spaceable.*

From the results due to Kitson and Timoney [7] we derive that $\ell_p^u(X) - \ell_p(X)$ for $p \geq 1$, and $\ell_p - \bigcup\limits_{0<q<p} \ell_q$ for $p > 1$, are spaceable. However, as is made clear in [7, Remark 2.2], their results are restricted to Fréchet spaces (see the Introduction). Next we extend the spaceability of $\ell_p^u(X) - \ell_p(X)$ and $\ell_p - \bigcup\limits_{0<q<p} \ell_q$ to the non-locally convex case:

Corollary 1.5. *$\ell_{m(s;p)}(X) - \ell_p(X)$ and $\ell_p^u(X) - \ell_p(X)$ are spaceable for $0 < p \leq s < \infty$ and every infinite-dimensional Banach space X. Hence $\ell_p^w(X) - \ell_p(X)$ is spaceable as well.*

Proof. By [9, Proposition 1.2(1)] we have that $\ell_{m(\infty;p)}(X) = \ell_p(X) \subseteq \ell_{m(s;p)}(X)$, and by [9, Theorem 2.1], $\ell_{m(s;p)}(X) \neq \ell_p(X)$. On the other hand, the identity operator on any infinite-dimensional Banach space fails to be absolutely p-summing for every $0 < p < \infty$ (the case $1 \leq p < \infty$ is well known, and the case $0 < p < 1$ follows from the fact that p-summing operators are q-summing whenever $p \leq q$). So $\ell_p^u(X) \neq \ell_p(X)$. As $\ell_{m(s;p)}(X)$ and $\ell_p^u(X)$ are invariant sequence spaces over X, the first assertion follows from Theorem 1.3. As $\ell_p^u(X) \subseteq \ell_p^w(X)$, the second assertion follows. $\square$

Before proving that $\ell_p - \bigcup\limits_{0<q<p} \ell_q$ is spaceable we have to check first that it is non-empty. Although we think this is folklore, we have not been able to find a reference in the literature. So, for the sake of completeness, we include a short proof, which was kindly communicated to us by M. C. Matos.

Lemma 1.6. *$\ell_p - \bigcup\limits_{0<q<p} \ell_q \neq \emptyset$ for every $p > 0$.*

Proof. Since $\left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty} \notin \ell_2$ and $\left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty} \in \ell_r$ for all $r > 2$, for each $(y_n)_{n=1}^{\infty} \in \ell_q$, $0 < q < 2$, it follows from Hölder's inequality that

$$\sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{n}} y_n \right| \leq \left\| \left(\frac{1}{\sqrt{n}} \right)_{n=1}^{\infty} \right\|_{q'} \cdot \left\| (y_n)_{n=1}^{\infty} \right\|_q < \infty.$$

Supposing that $\ell_2 = \bigcup\limits_{0<q<2} \ell_q$, we have that $\sum\limits_{n=1}^{\infty} \left| \frac{1}{\sqrt{n}} y_n \right| < \infty$ for every $(y_n)_{n=1}^{\infty} \in \ell_2$. So, consider, for each positive integer k, the continuous linear functional on ℓ_2 defined by $T_k\left((y_n)_{n=1}^{\infty}\right) = \sum\limits_{n=1}^{k} \frac{1}{\sqrt{n}} y_n$. As

$$\sup_{k \in \mathbb{N}} \left| T_k\left((y_n)_{n=1}^{\infty}\right) \right| = \sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{n}} y_n \right| < \infty$$

for each $(y_n)_{n=1}^{\infty} \in \ell_2$, by the Banach-Steinhaus Theorem we conclude that

$$T\left((y_n)_{n=1}^{\infty}\right) = \lim_k T_k\left((y_n)_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} y_n$$

defines a continuous linear functional on ℓ_2 and it follows that $\left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty} \in \ell_2$ - a contradiction which proves that there is $x \in \ell_2 - \bigcup\limits_{0<q<2} \ell_q$. So $\left(|x_n|^{\frac{2}{p}}\right)_{n=1}^{\infty} \in \ell_p - \bigcup\limits_{0<q<p} \ell_q$. $\square$

Corollary 1.7. $\ell_p - \bigcup\limits_{0<q<p} \ell_q$ *is spaceable for every* $p > 0$.

Proof. We know that ℓ_p is an invariant sequence space over $\mathbb{K}$ and from Lemma 1.6 we have $\ell_p - \bigcup\limits_{0<q<p} \ell_q \neq \emptyset$. The result follows from Theorem 1.3. $\square$

Remark 1.8. Theorem 1.3 can be applied in a variety of other situations. For example, for Lorentz spaces it applies to $\ell_{q,r} - \ell_p$ for $0 < p < q$ and $r > 0$, and to $\ell_{p,q} - \ell_p$ for $0 < p < q$. We believe that the usefulness of Theorem 1.3 is well established, so we refrain from giving further applications.

Although our results concern spaceability of complements of linear subspaces, the same technique gives the spaceability of sets that are not related to linear subspaces at all. Rewriting the proof of Theorem 1.3 we get:

Proposition 1.9. *Let* E *be an invariant sequence space over the Banach space* X. *Let* $A \subseteq E$ *be such that:*
(i) *For* $x \in E$, $x \in A$ *if and only if* $x^0 \in A$.
(ii) *If* $x = (x_j)_{j=1}^{\infty} \in A$ *and* $y = (y_j)_{j=1}^{\infty} \in E$ *is such that* $(\|y_j\|)_{j=1}^{\infty}$ *is a multiple of a subsequence of* $(\|x_j\|)_{j=1}^{\infty}$, *then* $y \in A$.
(iii) *There is* $x \in E - A$ *with* $x^0 \neq 0$.
Then $E - A$ *is spaceable.*

2 Norm-attaining operators

In this section we show that the technique used in the previous section can be used in a completely different context. Specifically, we extend a result from [11] concerning the lineability of the set of norm-attaining operators.

Given Banach spaces E and F and $x_0 \in E$ such that $\|x_0\| = 1$ (x_0 is said to be a *norm-one vector*), a continuous linear operator $u\colon E \longrightarrow F$ attains its norm at x_0 if $\|u(x_0)\| = \|u\|$. By $\mathcal{NA}^{x_0}(E;F)$ we mean the set of continuous linear operators from E to F that attain their norms at x_0.

In [11, Proposition 6] it is proved that if F contains an isometric copy of ℓ_q for some $1 \leq q < \infty$, then $\mathcal{NA}^{x_0}(E;F)$ is $\aleph_0$-lineable. We generalize this result showing that this set is c-lineable:

Proposition 2.1. *Let* E *and* F *be Banach spaces so that* F *contains an isometric copy of* ℓ_q *for some* $1 \leq q < \infty$, *and let* x_0 *be a norm-one vector in* E. *Then* $\mathcal{NA}^{x_0}(E;F)$ *is* c*-lineable.*

Proof. The beginning of the proof follows the lines of the proof of [11, Proposition 6]. It suffices to prove the result for $F = \ell_q$. Split $\mathbb{N}$ into

countably many infinite pairwise disjoint subsets $(A_k)_{k=1}^{\infty}$. For each positive integer k, write $A_k = \{a_1^{(k)} < a_2^{(k)} < \ldots\}$ and define

$$\ell_q^{(k)} := \{x \in \ell_q : x_j = 0 \text{ if } j \notin A_k\}\,.$$

Fix a non-zero operator $u \in \mathcal{NA}^{x_0}(E;F)$ and proceed as in the proof of [11, Proposition 6] to get a sequence $(u^{(k)})_{k=1}^{\infty}$ of operators belonging to $\mathcal{NA}^{x_0}(E;\ell_q^{(k)})$ such that $\|u^{(k)}(x)\| = \|u(x)\|$ for every k and every $x \in E$. By composing these operators with the inclusion $\ell_q^{(k)} \hookrightarrow \ell_q$ we get operators (and we keep the notation $u^{(k)}$ for the sake of simplicity) belonging to $\mathcal{NA}^{x_0}(E;\ell_q)$. For every $(a_k)_{k=1}^{\infty} \in \ell_1$,

$$\sum_{k=1}^{\infty} \|a_k u^{(k)}\| = \sum_{k=1}^{\infty} |a_k| \|u^{(k)}\| = \sum_{k=1}^{\infty} |a_k| \|u^{(k)}(x_0)\| = \|u(x_0)\| \sum_{k=1}^{\infty} |a_k| < \infty,$$

so the map

$$T\colon \ell_1 \longrightarrow \mathcal{L}(E;\ell_q) \ \ , \ \ T((a_k)_{k=1}^{\infty}) = \sum_{k=1}^{\infty} a_k u^{(k)}$$

is well-defined. It is clear that T is linear and injective. Hence $T(\ell_1)$ is a $\mathfrak{c}$-dimensional subspace of ℓ_q. Since the supports of the operators $u^{(k)}$ are pairwise disjoint, $T(\ell_1) \subseteq \mathcal{NA}^{x_0}(E;\ell_q)$. $\square$

References

[1] R. M. Aron, V. I. Gurariy, J. B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc. **133** (2005) 795–803.

[2] R. M. Aron, F. J. García-Pacheco, D. Pérez-García, J. B. Seoane-Sepúlveda, *On dense-lineability of sets of functions on* $\mathbb{R}$, Topology **48** (2009) 149–156.

[3] L. Bernal-González, *Dense-lineability in spaces of continuous functions*, Proc. Amer. Math. Soc. **136** (2008) 3163–3169.

[4] G. Botelho, D. Diniz, D. Pellegrino and E. Teixeira, *A note on lineability*, arXiv:0905.2677 (2009).

[5] N. J. Kalton, *The basic sequence problem*, Studia Math. **116** (1995), 167-187.

[6] N. J. Kalton, *Quasi-Banach spaces*, Handbook of the Geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1099–1130.

[7] D. Kitson and R. Timoney, *Some applications of operator ranges*, preprint.

[8] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, Springer, 1996.

[9] M. C. Matos, *Mappings between Banach spaces that send mixed summable sequences into absolutely summable sequences*, J. Math. Anal. Appl. **297** (2004), 833-851.

[10] G. Muñoz-Fernandez, N. Palmberg, D. Puglisi e J. B. Seoane-Sepúlveda, *Lineability in subsets of measure and function spaces*, Linear Algebra Appl. **428** (2008), 2805-2812.

[11] D. Pellegrino and E. Teixeira, *Norm optimization problem for linear operators in classical Banach spaces*, Bull. Braz. Math. Soc. **40** (2009), 417-431.

[12] A. Pietsch, *Operator ideals*, North-Holland Publishing Company, 1980.

[13] D. Puglisi, J. B. Seoane-Sepúlveda, *Bounded linear non-absolutely summing operators*, J. Math. Anal. Appl. **338** (2008), 292-298.

[14] A. Wilansky, *Semi-Fredholm maps of FK spaces*, Math. Z. **144** (1975), 9-12.

[Geraldo Botelho and Vinícius V. Fávaro] Faculdade de Matemática, Universidade Federal de Uberlândia, 38.400-902 - Uberlândia, Brazil, e-mails: botelho@ufu.br, vvfavaro@gmail.com.

[Diogo Diniz] UAME-UFCG, Caixa Postal 10044 - 58.109-970, Campina Grande, Brazil, e-mail: diogodme@gmail.com.

[Daniel Pellegrino] Departamento de Matemática, Universidade Federal da Paraíba, 58.051-900 - João Pessoa, Brazil, e-mail: dmpellegrino@gmail.com.